arXiv:0903.4970v1 [math.CV] 28 Mar 2009

ASYMPTOTICS OF THE HOLE PROBABILITY FOR ZEROS OF RANDOM ENTIRE FUNCTIONS

ALON NISHRY

Abstract. Consider the random entire function

$$f(z)=\sum_{n=0}^{\infty}\phi_n\,\frac{z^n}{\sqrt{n!}},\tag{$*$}$$

where the ϕ_n are i.i.d. standard complex Gaussian variables. The zero set of this function is distinguished by invariance of its distribution with respect to the isometries of the plane.

We study the probability $P_H(r)$ that f has no zeroes in the disk $\{|z|<r\}$ (hole probability). Improving a result of Sodin and Tsirelson, we show that

$$\log P_H(r)=-\frac{3e^2}{4}\cdot r^4+o(r^4)$$

as $r\to\infty$. The proof does not use distribution invariance of the zeros, and can be extended to other Gaussian Taylor series.

If instead of Gaussians we take Rademacher or Steinhaus random variables ϕ_n, we get a very different result. There exists r_0 so that every random function of the form $(*)$ with Rademacher or Steinhaus coefficients must vanish in the disk $\{|z|<r_0\}$.

1. Introduction

Consider the following random entire function

$$f(z)=\sum_{n=0}^{\infty}\phi_n a_n z^n,\tag{1.1}$$

where $a_n=(n!)^{-1/2}$ and ϕ_n are independent standard complex Gaussian random variables (i.e., each ϕ_n has the density function $\pi^{-1}\cdot\exp\left(-|z|^2\right)$ with respect to Lebesgue measure on $\mathbb{C}$). The random zero set of this function is known to be distribution invariant with respect to isometries of the plane. Furthermore, this is the only Gaussian entire function with distribution invariant zeros (see [ST1], and the forthcoming book [BKPV] for details and discussion).

One of the interesting characteristics of the random zero process $f^{-1}\{0\}$ is the asymptotic decay of the event where f has no zeros

inside the disk $\{|z| \le r\}$ when r is large. Since the decay rate is known to be exponential, we use the notation

$$p_H(z) = \log^- P_H(r) = \log^- \mathbb{P}\left(f(z) \ne 0 \text{ for } |z| \le r\right).$$

In the paper [ST3], Sodin and Tsirelson showed that for $r \ge 1$,

$$c_1 r^4 \le p_H(r) \le c_2 r^4 \tag{1.2}$$

with some positive numerical constants c_1 and c_2. This result was extended in different directions by Ben Hough [BH], Krishnapur [K], Zrebiec [Zr1, Zr2] and Shiffman, Zelditch and Zrebiec [SZZ].

In [ST3], Sodin and Tsirelson raised the question whether the limit

$$\lim_{r\to\infty} \frac{p_H(r)}{r^4}$$

exists and what is its value ? Our main result answers this question (and estimates the remainder):

Theorem 1. *For r large enough*

$$p_H(r) = \frac{3e^2}{4} \cdot r^4 + \mathcal{O}\left(r^{18/5}\right). \tag{1.3}$$

The constant $\frac{3e^2}{4}$ arrives as follows. We introduce the function

$$S(r) = \log \prod_{\{n\,:\,a_n r^n \ge 1\}} \left(a_n r^n\right)^2 = 2 \cdot \sum_{\{n\,:\,a_n r^n \ge 1\}} \log\left(a_n r^n\right),$$

and prove that

$$p_H(r) = S(r) + \mathcal{O}\left(r^{18/5}\right), \qquad r \to \infty. \tag{1.4}$$

Then it is easy to see that

$$S(r) = \frac{3e^2}{4} \cdot r^4 + \mathcal{O}\left(r^2 \log r\right), \qquad r \to \infty.$$

Actually, it is plausible that estimate (1.3) holds with a better estimate of the remainder, for instance, with $\mathcal{O}\left(r^{2+\epsilon}\right)$ with any $\epsilon > 0$.

The proof of the upper bound in (1.4) is similar to the proof of the upper bound in (1.2) given in [ST3], the only difference is that our estimates are slightly more accurate (note that in [ST3] this is considered to be the lower bound). The proof of the lower bound in (1.4) combines techniques from [ST3] with direct estimates of probability of some events in high-dimensional linear spaces with Gaussian measure. Note that somewhat similar ideas were used in [SZZ].

In contrast to [ST3], our proof of the lower bound in (1.4) does not use distribution invariance of the zero set of f, and our main result can be extended to other Gaussian entire functions of the form (1.1) with

regular sequence of the coefficients a_n. For instance, one may consider Gaussian Mittag-Leffler functions

$$f(z) = \sum_{n=0}^{\infty} \phi_n \cdot \frac{z^n}{\Gamma(\alpha n + 1)}$$

with $\alpha > 0$. In this case, the corresponding function $S(r)$ has the asymptotics

$$S(r) = \frac{1}{2\alpha} r^{2/\alpha} + \mathcal{O}_\alpha \left(r^{1/\alpha} \log r \right)$$

and then only minor modifications in the proof of Theorem 1 are needed to show that

$$p_H(r) = S(r) + \mathcal{O}_\alpha \left(r^{9/5\alpha} \right).$$

One can ask what happens when the i.i.d. coefficients ζ_n in (1.1) are not Gaussian ? The following deterministic result shows that the situation might be very different.

Theorem 2. *Let $K \subset \mathbb{C}$ be a compact set and $0 \notin K$. Suppose that $\phi_n \in K$ for each n, and that $a_n = (n!)^{-1/2}$. Then there exists $r_0(K) < \infty$ so that $f(z)$ must vanish somewhere in the disk $\{|z| \le r_0(K)\}$.*

The idea of the proof is very simple. A standard compactness argument shows that if the result is wrong then there exists an entire function f of the form (1.1) with $\phi_n \in K$ and $a_n = (n!)^{-1/2}$ that does not vanish on $\mathbb{C}$. Since f is an entire function of order 2, it equals $\exp(\alpha z^2 + \beta z + \gamma)$ with complex constants α, β, γ. Then it is not difficult to verify that the Taylor coefficients of that function cannot be equal to $\phi_n / \sqrt{n!}$ with $\zeta_n \in K$.

Acknowledgement. First of all I would like to thank my advisor Mikhail Sodin for his more than substantial involvement in writing this paper, and supporting me in general. Starting with the subject, continuing with technical issues, and especially in the presentation, his fingerprints are found everywhere in this paper. I would also like to thank Fedor Nazarov for contributing his idea regarding the second theorem.

2. Notations and Elementary Estimates

In what follows we frequently use that if w is a standard Gaussian random variable, then,

$$\mathbb{P}(|w| \ge \lambda) = \exp(-\lambda^2), \tag{2.1}$$

and for

$$\lambda \le 1,$$

$$\mathbb{P}\left(|w| \le \lambda\right) \in \left[\frac{\lambda^2}{2}, \lambda^2\right]. \tag{2.2}$$

We denote by $r\mathbb{D}$the disk $\{z\,:\,|z|<r\}$ and by $r\mathbb{T}$ its boundary $\{z\,:\,|z|=r\}$. The letter C denotes positive numerical constant (which can change between lines).

In what follows, we use several elementary estimates, we skip their proofs.

Lemma 3. *The sequence $a_n r^n$ has a local maximum only in the interval $n \in \{\lceil r^2-1\rceil, \lfloor r^2 \rfloor\}$.*

Using Striling's approximation we have

Lemma 4. *For all $n \ge 1$ we have*

$$\frac{1}{\sqrt{3n}}\left(\frac{e}{n}\right)^{\frac{n}{2}} \le a_n \le \left(\frac{e}{n}\right)^{\frac{n}{2}}, \tag{2.3}$$

moreover for $n \in \{1, \dots, \lfloor er^2 \rfloor\}$

$$\frac{1}{3r}\cdot\left(a_n r^n\right)^{-1} \le 1 \tag{2.4}$$

and for $n \ge er^2$

$$\log\left(a_n r^n\right) \le -\frac{1}{2}\left(n - er^2\right). \tag{2.5}$$

Since by Lemma 3 we see the sequence $a_n r^n$ is unimodal in $[0, \lfloor er^2 \rfloor]$, it is easy to estimate $S(r)$ with a corresponding integral and get

Lemma 5. *We have*

$$S(r) = \frac{3e^2}{4}\cdot r^4 + \mathcal{O}\left(r^2 \log r\right).$$

3. Upper Bound for $p_H(r)$

In this section, we show that for r large enough,

$$p_H(r) \le S(r) + \mathcal{O}\left(r^2 \log r\right).$$

Proof. Denote by Ω_r the following event

$$\begin{array}{lll} \text{(i)} & |\phi_0| \ge 2r & , \\ \text{(ii)} & |\phi_n| \le \frac{1}{3r}\cdot\left(a_n r^n\right)^{-1} & n \in \{1,\dots,\lfloor er^2 \rfloor\}, \\ \text{(iii)} & |\phi_n| \le \exp\left(\frac{n-er^2}{4}\right) & n \ge \lfloor er^2 \rfloor + 1. \end{array}$$

We prove that if r is large enough and the event Ω_r occurs, then $f(z) \neq 0$ inside $r\mathbb{D}$, and that

$$\log \mathbb{P}(\Omega_r) \geq -S(r) - \mathcal{O}\left(r^2 \log r\right).$$

Note that

$$|f(z)| \geq |\phi_0| - \sum_{n=1}^{\infty} |\phi_n| a_n r^n. \tag{3.1}$$

First, estimate the sum

$$\sum_{n=1}^{\lfloor er^2 \rfloor} |\phi_n| a_n r^n \leq \sum_{n=1}^{\lfloor er^2 \rfloor} \frac{1}{3r} \leq r.$$

To bound the tail we use (2.5),

$$\begin{aligned} \sum_{n \geq \lfloor er^2 \rfloor + 1} |\phi_n| a_n r^n &\leq \sum_{\lfloor er^2 \rfloor + 1}^{\infty} \exp\left(\frac{n - er^2}{4} - \frac{1}{2}\left(n - er^2\right)\right) \\ &\leq \sum_{n=0}^{\infty} \exp(-n/2) = \mathcal{O}(1). \end{aligned}$$

From (3.1), we have

$$|f(z)| \geq 2r - r - \mathcal{O}(1) > 0,$$

for r large enough. We see that $f(z) \neq 0$ inside $r\mathbb{D}$.

Now we estimate the probability of Ω_r using (2.1) and (2.2). First,

$$\mathbb{P}((\mathrm{i})) = \exp\left(-4r^2\right).$$

For $n \geq \lfloor er^2 \rfloor + 1$, we have

$$\mathbb{P}\left((\mathrm{iii})_n\right) = 1 - \exp\left(-\exp\left(\frac{n - er^2}{2}\right)\right).$$

That is,

$$\mathbb{P}((\mathrm{iii})) = \prod_{n \geq \lfloor er^2 \rfloor + 1} \mathbb{P}\left((\mathrm{iii})_n\right) = \exp\left(\sum_{n \geq \lfloor er^2 \rfloor + 1} \log \mathbb{P}\left((\mathrm{iii})_n\right)\right).$$

Taking logarithm of $\mathbb{P}\left((\mathrm{iii})_n\right)$, we can see that we have the following estimate for r large enough,

$$\log \mathbb{P}\left((\mathrm{iii})_n\right) \geq -\exp\left(-\frac{n - er^2}{2}\right),$$

so $\mathbb{P}\left((\mathrm{iii})\right)$ is larger than some constant, which does not depend on r. For the term $\mathbb{P}\left((\mathrm{ii})\right)$, recalling (2.4), we use the estimate

$$\mathbb{P}\left((\mathrm{ii})_n\right) \geq \frac{(a_n r^n)^{-2}}{18r^2},$$

and get

$$\mathbb{P}\left((\mathrm{ii})\right) \geq \prod_{n=0}^{\lfloor er^2 \rfloor} \frac{(a_n r^n)^{-2}}{18r^2} = \exp\left(-S(r) - \lfloor er^2 \rfloor \cdot \log 18r^2\right).$$

Since $\mathbb{P}\left(\Omega_r\right) = \mathbb{P}\left((\mathrm{i})\right)\mathbb{P}\left((\mathrm{ii})\right)\mathbb{P}\left((\mathrm{iii})\right)$, we get the required result:

$$p_H(r) \leq -\log \mathbb{P}\left(\Omega_r\right) \leq S(r) + \mathcal{O}\left(r^2 \log r\right).$$

□

4. Lower Bound for $p_H(r)$

In this section we show that for r large enough

$$p_H(r) \geq S(r) - Cr^{18/5}.$$

Define $M(r) = \max\limits_{|z|\leq r} |f(z)|$, we start by studying the deviations of $\log M(r)$ from the mean $\frac{1}{2}r^2$. Then we consider large deviations of the expression $\int\limits_{r\mathbb{T}} \log|f(z)|\,dm$, where m is the normalized angular measure on $r\mathbb{T}$. Finally, we use the fact that if $n(r) = 0$ then $\log|f(z)|$ is a harmonic function inside $r\mathbb{D}$ to get the result.

4.1. Large deviations for $\log M(r)$.

We use the first part of Lemma 1 in the paper [ST3] as

Lemma 6. *Given $\sigma > 0$, we have for r large enough*

$$\log \mathbb{P}\left(\frac{\log M(r)}{\frac{1}{2}r^2} \geq 1 + \sigma\right) \leq -\exp\left(\sigma r^2\right).$$

In the other direction we have

Lemma 7. *We have the following estimate for the lower bound of $M(r)$*

$$\log \mathbb{P}\left(M(r) \leq 1\right) \leq -S(r).$$

Proof. Suppose $\log|f(z)| \leq 0$ in $r\mathbb{D}$, then using Cauchy's estimate for the coefficients of $f(z)$ we can get an estimate to the probability of this event. We have

$$|\phi_n| a_n r^n \leq M(r) \leq 1$$

or

$$|\phi_n| \le (a_n r^n)^{-1}.$$

The probability of each event, for $n \in \{0, \dots, \lfloor er^2 \rfloor\}$ is bounded by (using Lemma 4)

$$\mathbb{P}\left(|\phi_n| \le (a_n r^n)^{-1}\right) \le (a_n r^n)^{-2}.$$

We get

$$\mathbb{P}\left(M(r) \le 1\right) \le \prod_{n=0}^{\lfloor er^2 \rfloor} (a_n r^n)^{-2} = \exp\left(-S(r)\right).$$

□

4.2. **Discretization of the logarithmic integral.** In this section $\delta \in (0,1)$, $N = \lfloor er^2 \rfloor$, $\kappa = 1 - \delta^{1/2}$ and the points $\{z_j\}_{j=0}^{N-1}$ are equally distributed on $\kappa r\mathbb{T}$, that is

$$z_j = \kappa r \exp\left(\frac{2\pi i j}{N}\right).$$

Also m is the normalized angular measure on $r\mathbb{T}$. Under this conditions we have

Lemma 8. *Outside an exceptional set of probability at most*

$$2\exp\left(-S(\kappa r)\right)$$

we have

$$\frac{1}{N}\sum_{j=0}^{N-1} \log|f(z_j)| \le \int\limits_{r\mathbb{T}} \log|f|\, dm + \frac{C}{\delta^2} \tag{4.1}$$

with C a positive numerical constant.

Proof. Denote by $P_j(z) = P(z, z_j)$ the Poisson kernel for the disk $r\mathbb{D}$,$|z| = r$, $|z_j| < r$. Since $\log|f|$ is a subharmonic function we have

$$\begin{aligned}
\frac{1}{N}\sum_{j=0}^{N-1} \log|f(z_j)| &\le \int\limits_{r\mathbb{T}} \left(\frac{1}{N}\sum_{j=0}^{N-1} P_j\right) \log|f|\, dm \\
&= \int\limits_{r\mathbb{T}} \log|f|\, dm + \int\limits_{r\mathbb{T}} \left(\frac{1}{N}\sum_{j=0}^{N-1} P_j - 1\right) \log|f|\, dm.
\end{aligned}$$

The last expression can be estimated by

(4.2)

$$\int\limits_{r\mathbb{T}} \left(\frac{1}{N}\sum_{j=0}^{N-1} P_j - 1\right) \log|f|\, dm \le \max_{z\in r\mathbb{T}} \left|\frac{1}{N}\sum_{j=0}^{N-1} P_j - 1\right| \cdot \int\limits_{r\mathbb{T}} \left|\log|f|\right|\, dm.$$

For the first factor in the RHS of (4.2), we start with

$$\int_{\kappa r\mathbb{T}} P(z,\omega)\, dm(\omega) = 1,$$

and then split the circle $\kappa r\mathbb{T}$ into a union of N disjoint arcs I_j of equal angular measure $\mu(I_j) = \frac{1}{N}$ centered at the z_j's. Then

$$1 = \frac{1}{N}\sum_{j=0}^{N-1} P(z,z_j) + \sum_{j=0}^{N-1}\int_{I_j} \left(P(z,\omega) - P(z,z_j)\right)\, dm(\omega),$$

and

$$\begin{aligned} |P(z,\omega) - P(z,z_j)| &\le \max_{\omega\in I_j}|\omega - z_j| \cdot \max_{z,\omega}|\nabla_\omega P(z,\omega)| \\ &\le \frac{2\pi r}{N}\cdot\frac{Cr}{(r-|\omega|)^2} \le \frac{C}{\delta N}. \end{aligned}$$

For the second factor on the RHS of (4.2), using Lemma 7, we may suppose that there is a point $a \in \kappa r\mathbb{T}$ such that $\log|f(a)| \ge 0$ (discarding an exceptional event of probability at most $\exp(-S(\kappa r))$). Then we have

$$0 \le \int_{r\mathbb{T}} P(z,a)\log|f(z)|\, dm(z),$$

and hence

$$\int_{r\mathbb{T}} P(z,a)\log^-|f(z)|\, dm(z) \le \int_{r\mathbb{T}} P(z,a)\log^+|f(z)|\, dm(z).$$

For $|z| = r$ and $|a| = \kappa r$ we have,

$$\frac{\delta^{\frac{1}{2}}}{2} \le \frac{1-(1-\delta^{\frac{1}{2}})}{1+(1-\delta^{\frac{1}{2}})} \le P(z,a) \le \frac{1+(1-\delta^{\frac{1}{2}})}{1-(1-\delta^{\frac{1}{2}})} \le \frac{2}{\delta^{\frac{1}{2}}}.$$

By Lemma 6, outside a very small exception set (of the order $\exp(-\exp(r^2))$), we have

$$\int_{r\mathbb{T}} \log^+|f|\, d\mu \le r^2,$$

and therefore

$$\int_{r\mathbb{T}} \log^-|f|\, d\mu \le \frac{Cr^2}{\delta}.$$

Finally

$$\int_{r\mathbb{T}} |\log|f||\, d\mu \le \frac{Cr^2}{\delta}.$$

Using the fact that $N = \lfloor er^2 \rfloor$, we see that the lemma is proved. $\square$

4.3. **Deviation from the logarithmic integral.** If we use the notation $\zeta_j = f(z_j)$, we know that the Gaussian random variables $\{\zeta_j\}_{j=0}^{N-1}$ have a multivariate complex Gaussian distribution, with covariance matrix Σ, where

$$\begin{aligned} \Sigma_{ij} &= \mathrm{Cov}(\zeta_i, \zeta_j) = \mathrm{Cov}(f(z_i), f(z_j)) \\ &= \mathbb{E}(f(z_i)\overline{f(z_j)}) = \sum_{n=0}^{\infty} a_n^2 z_i \bar{z}_j = \exp\left(z_i \bar{z}_j\right). \end{aligned}$$

We also know that the density function of this distribution is

$$\zeta \mapsto \frac{1}{\pi^n \cdot \det \Sigma} \cdot \exp(-\zeta^* \Sigma^{-1} \zeta).$$

We introduce the sets

$$\mathcal{A}' = \left\{ \zeta \ : \ \prod_{j=0}^{N-1} |\zeta_j| \le \exp\left(4N \cdot \log r + C\delta^{-2} \cdot r^2\right) \right\} \tag{4.3}$$

and

(4.4)

$$\mathcal{A} = \left\{ \zeta \ : \ \zeta \in \mathcal{A}' \text{ and } |\zeta_j| = |f(z_j)| \le \exp\left(2r^2\right), \quad 0 \le j \le N-1 \right\}$$

and denote by $\mathcal{B}$ the set where estimate (4.1) in Lemma 8 holds. Using these notations we get the simple

Lemma 9.

$$\mathbb{P}\left(\int_{r\mathbb{T}} \log |f(z)| \, d\mu \le 4 \log r \right) \le \mathbb{P}\left(\mathcal{A}\right) + \mathbb{P}\left(\mathcal{B}^c\right) + \mathbb{P}\left(\mathcal{A}' \backslash \mathcal{A}\right).$$

Proof. We start by discarding the exceptional set in Lemma 8, this adds the term $\mathbb{P}\left(\mathcal{B}^c\right)$. Now we can assume that

$$\frac{1}{N} \sum_{j=0}^{N-1} \log |f(z_j)| \le \int_{r\mathbb{T}} \log |f| \, dm + \frac{C}{\delta^2},$$

or

$$\prod_{j=0}^{N-1} |\zeta_j| \le \exp\left(N \cdot \int_{r\mathbb{T}} \log |f| \, dm + \frac{C}{\delta^2} \cdot N \right).$$

In terms of probabilities we can write

$$\mathbb{P}\left(\int_{r\mathbb{T}} \log |f(z)| \, dm \le 4 \log r \right) \le \mathbb{P}\left(\mathcal{B}^c\right) + \mathbb{P}\left(\mathcal{A}'\right),$$

and since

$$\mathbb{P}\left(\mathcal{A}'\right) = \mathbb{P}\left(\mathcal{A}\right) + \mathbb{P}\left(\mathcal{A}' \backslash \mathcal{A}\right),$$

we get the required result. □

Before we continue, we need two asymptotic estimates.

Lemma 10. *Let Σ be the covariance matrix defined above. We have the following estimate*

$$\log\left(\det\Sigma\right) \geq S(\kappa r).$$

Proof. Notice that we can represent Σ in the following form

$$\Sigma = V \cdot V^*$$

where

$$V = \begin{pmatrix} a_0 & a_1 \cdot z_1 & \dots & a_N \cdot z_1^N & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \\ a_0 & a_1 \cdot z_N & \dots & a_N \cdot z_N^N & \dots \end{pmatrix}.$$

By the Cauchy-Binet formula we have

$$\det\ \Sigma = |\Sigma| = \sum_t |m_t(V)|^2,$$

where the sum is taken over all principal minors $m_t(V)$ of the matrix V.

In our case the following minor is sufficient for the estimates

$$\begin{aligned} |\Sigma| &\geq \begin{vmatrix} a_1 z_1 & a_2 z_1^2 & \dots & a_N z_1^N \\ \vdots & \vdots & \vdots & \vdots \\ a_1 z_N & a_2 z_N^2 & \dots & a_N z_N^N \end{vmatrix}^2 \\ &= \prod_{n=1}^{N} a_n^2 \cdot \prod_{i=1}^{N} |z_i|^2 \cdot \prod_{1\leq i\neq j\leq N} |z_i - z_j| \\ &= \Pi_1 \cdot \Pi_2 \cdot \Pi_3. \end{aligned}$$

It is clear that

$$\Pi_2 = (\kappa r)^{2N}.$$

The z_i's are the roots of the equation $z^N = (\kappa r)^N$, denoting $z_1 = \kappa r$ we get

$$\prod_{i=2}^{N} (z_1 - z_i) = N \cdot (\kappa r)^{N-1},$$

and

$$\Pi_3 = \prod_{1\leq i\neq j\leq N} |z_i - z_j| = \left(\prod_{i=2}^{N} |z_1 - z_i| \right)^N = (\kappa r)^{N(N-1)} \cdot N^N.$$

We now "collect" the product of the κr's and rewrite it as $(\kappa r)^{N(N-1)} = \prod_{n=1}^{N-1} (\kappa r)^{2n}$ and get

$$\det \Sigma \geq \prod_{n=1}^{N} a_n^2 (\kappa r)^{2n}.$$

Using the fact that $N = \lfloor er^2 \rfloor$, we have

$$\det \Sigma \geq \exp(S(\kappa r)).$$

□

We denote by I the following quantity

$$I = \pi^{-N} \cdot \mathrm{vol}_{\mathbb{C}^N}(\mathcal{A}). \tag{4.5}$$

We use the following lemma to estimate I,

Lemma 11. *Set $s > 0$, $t > 0$ and $N \in \mathbb{N}^+$, such that $\log\left(t^N/s\right) \geq N$. Denote by $\mathcal{C}_N$ the following set*

$$\mathcal{C}_N = \mathcal{C}_N(t,s) = \left\{ r = (r_1, \ldots, r_N) \ : \ 0 \leq r_j \leq t, \ \prod_1^N r_j \leq s \right\}.$$

Then

$$\mathrm{vol}_{\mathbb{R}^N}(\mathcal{C}_N) \leq \frac{s}{(N-1)!} \log^N\left(t^N/s\right).$$

Proof. We will find an expression for the volume using induction. First we define for $k \geq 1$

$$V_k(t,s) = \mathrm{vol}_{\mathbb{R}^k}(\mathcal{C}_k(t,s)).$$

We notice that if $s \geq t^k$ then $V_k(t,s) = t^k$. Now we can write

$$V_k(t,s) = \int_0^t \left(\int_0^t \ldots \int_0^t \chi\left(\prod_2^k r_j \leq s/x\right) dr_2 \ldots dr_k \right) dx,$$

where χ is the characteristic function of the set $\left\{r \ : \ \prod_2^k r_j \leq s/x\right\}$, if $s < t^k$ we can rewrite this expression as

$$\begin{aligned} V_k(t,s) &= \int_0^a V_{k-1}\left(t, \frac{s}{x}\right) dx \\ &= \int_0^{s/t^{k-1}} V_{k-1}\left(t, \frac{s}{x}\right) dx + \int_{s/t^{k-1}}^t V_{k-1}\left(t, \frac{s}{x}\right) dx = I_1 + I_2. \end{aligned}$$

For the first integral we have $x < s/t^{k-1}$ or $s/x > t^{k-1}$ and so $V_{k-1}\left(t, \frac{s}{x}\right) = t^{k-1}$, meaning $I_1 = s$. We can now prove by induction

$$V_k(t,s) = \begin{cases} t^k & s \ge t^k, \\ \sum_{m=0}^{k-1} \frac{s}{m!} \cdot \left(\log\left(t^k/s\right)\right)^m & s < t^k. \end{cases}$$

For $k = 1$ this is trivial, now using the expression above we have,

$$\begin{aligned} V_k(t,s) &= s + \int\limits_{s/t^{k-1}}^{t} \sum_{m=0}^{k-2} \frac{s}{x \cdot m!} \cdot \left(\log\left(xt^{k-1}/s\right)\right)^m \, dx \\ &= s + \sum_{m=0}^{k-2} \left[\frac{s}{(m+1)!} \log^{m+1}\left(\frac{t^{k-1}x}{s}\right)\Big|_{x=s/t^{k-1}}^{t} \right] \\ &= \sum_{m=0}^{k-1} \frac{s}{m!} \cdot \left(\log\left(t^k/s\right)\right)^m . \end{aligned}$$

We conclude that

$$\mathrm{vol}_{\mathbb{R}^N}(\mathcal{C}_N) = \sum_{m=0}^{N-1} \frac{s}{m!} \cdot \left(\log\left(t^N/s\right)\right)^m .$$

Since $\log\left(t^N/s\right) \ge N$, we can approximate the integral from above by

$$\begin{aligned} V_N(t,s) &= \sum_{m=0}^{N-1} \frac{s}{m!} \cdot \log^m\left(t^N/s\right) \le s \cdot \sum_{m=0}^{N-1} \frac{\log^m\left(t^N/s\right)}{m!} \cdot \frac{\log^{N-m}\left(t^N/s\right)}{(m+1)\cdot \ldots \cdot N} \\ &\le \frac{s}{(N-1)!} \log^N\left(t^N/s\right). \end{aligned}$$

This finishes the proof. □

Now we have as an almost immediate

Corollary 12. *For r large enough and $\delta \ge r^{2-\epsilon}$, we have*

$$\log I \le C\left(\log r + \delta^{-2}\right) r^2.$$

Proof. We recall that

$$\mathcal{A} = \left\{ \zeta \; : \; \begin{array}{c} |\zeta_j| = |f(z_j)| \le \exp\left(2r^2\right), \quad 0 \le j \le N-1 \\ \text{and} \\ \prod_{j=0}^{N-1} |\zeta_j| \le \exp\left(4N \cdot \log r + C\delta^{-2} \cdot r^2\right) \end{array} \right\}.$$

To shorten the expressions we use $s = \exp\left(4N \cdot \log r + C\delta^{-2} \cdot r^2\right)$ and $t = \exp\left(2r^2\right)$. We notice that under the assumptions that we made and for r large enough $\log\left(t^N/s\right) \ge N$ (remember that $N = \lfloor er^2 \rfloor$). We want to translate the integral into an integral in $\mathbb{R}^N$, using the change

of variables $\zeta_j = r_j \cos(\theta_j) + i r_j \sin(\theta_j)$. Integrating out the variables θ_j, we get $I' = 2^N \int_{\mathcal{C}} \prod r_j \, dr$, where the new domain is

$$\mathcal{C} = \left\{ r = (r_1, \ldots, r_N) \ : \ 0 \le r_j \le t, \prod_{j=1}^{N} r_j \le s \right\}.$$

We can find an explicit expression for this integral, but, instead we will simplify it even more to

$$I' \le 2^N \cdot s \cdot \mathrm{vol}_{\mathbb{R}^N}(\mathcal{C}) \tag{4.6}$$

Now we can use the previous lemma, and get (for r large enough)

$$\begin{aligned} I' &\le \frac{N \cdot 2^N \cdot s^2}{N!} \cdot \log^N \left(t^N/s\right) \\ &\le \frac{s^2 \cdot e^{2N}}{N^N} \cdot \log^N \left(t^N/s\right) \\ &= \exp\left(2\log s + N \log\log t + 2N - N \log\log s\right) \\ &\le \exp\left(2 \log s + N \log\log t\right). \end{aligned}$$

Recalling the definitions of s and t, we finally get

$$\log I' \le 8N \cdot \log r + C_1 r^2 \delta^{-2} + C_2 r^2 \log r \le C\left(\log r + \delta^{-2}\right) r^2.$$

$\square$

We now continue to estimate probabilities of the events $\mathcal{A}$ and $\mathcal{A}'$ introduced in (4.4) and (4.3).

Lemma 13. *We have the following estimates:*

$$\mathbb{P}\left(\mathcal{A}' \backslash \mathcal{A}\right) \le \exp\left(-\exp\left(r^2\right)\right)$$

and

$$\mathbb{P}\left(\mathcal{A}\right) \le \exp\left(-S(\kappa r) + C\left(\log r + \delta^{-2}\right) r^2\right).$$

Proof. The first estimate in a trivial consequence of Lemma 6. For the second we need to estimate the integral

$$I' = \int\limits_{\mathcal{A}} \frac{1}{\pi^N \cdot |\Sigma|} \cdot \exp\left(-\zeta^* \Sigma^{-1} \zeta\right) \, d\zeta.$$

We can use the crude estimate

$$I' \le \frac{1}{|\Sigma|} \cdot \int\limits_{\mathcal{A}} \frac{1}{\pi^N} \, d\zeta,$$

using the previous two lemmas, we get the result. $\square$

4.4. **Lower bound for p_H.** We collect all the previous results into this

Lemma 14. *For r large enough*

$$p_H(r) \geq S(\kappa r) - C\left(\log r + \delta^{-2}\right) r^2, \tag{4.7}$$

Proof. Suppose that $f(z)$ has no zeros inside $r\mathbb{D}$, then

$$\int_{r\mathbb{T}} \log|f(z)|\, dm = \log|f(0)|.$$

We can use the fact that $\log|f(0)|$ cannot be too large, in fact

$$\mathbb{P}\left(\log|f(0)| \geq 4\log r\right) = \mathbb{P}\left(|\phi_0| \geq r^4\right) \leq \exp\left(-r^8\right).$$

Now using this result combined with Lemma 9 and Lemma 13, we can bound the probability of this event by

$$\exp\left(-r^8\right) + 2\exp\left(-S(\kappa r)\right) + \exp\left(-S(\kappa r) + C\left(\log r + \delta^{-2}\right) r^2\right)$$

that is (4.7). □

To finish the proof of the lower bound for $p_H(r)$, we recall that $\kappa = 1 - \delta^{1/2}$ and select $\delta = r^{-\alpha}$ with $0 < \alpha < 2$, then $\kappa = 1 - r^{-\alpha/2}$. We have to minimize the asymptotics of the expression

$$-S(\kappa r) + C\left(\log r + \delta^{-2}\right) r^2.$$

Simple calculations lead to the equality $2 + 2\alpha = 4 - \alpha/2$, or $\delta = r^{-4/5}$. We finally get (for r large enough)

$$\begin{aligned} p_H(r) &\geq \frac{3e^2}{4} \cdot \left(r \cdot \left(1 - r^{-2/5}\right)\right)^4 - Cr^{18/5} \\ &\geq \frac{3e^2}{4} \cdot r^4 - Cr^{18/5} = S(r) - Cr^{18/5}. \end{aligned}$$

5. Proof of The Second Theorem

Suppose that the theorem is false, that is, there is a sequence of entire functions

$$f_k(z) = \sum_{n=0}^{\infty} \phi_{n,k} \cdot \frac{z^n}{\sqrt{n!}}, \qquad \phi_{n,k} \in K,$$

and a sequence $r_k \to \infty$ so that f_k does not vanish in $r_k\mathbb{D}$. Since K is a compact set, we can find a subsequence, also denoted by $\{f_k\}$, such that $\phi_{n,k} \to \phi_n$ for each $n \in \mathbb{N}$. It easy to see that the sequence $\{f_k\}$ converges locally uniformly to a limiting function f. Since $0 \notin K$, the limiting function f is not identically zero. Now, using Hurwitz theorem (see [Ahl, pg. 178]), f does not vanish in any disk $r_k\mathbb{D}$; i.e. is does not vanish in the whole complex plane.

By known formulas for the order and type of entire functions by its Taylor coefficients (see, for instance [Lev, pg. 6]) f has order 2 and type $\frac{1}{2}$. Since it does not vanish on $\mathbb{C}$, by Hadamard theorem, $f(z) = \exp\left(\alpha z^2 + \beta z + \gamma\right)$, with complex constants α, β, γ; $|\alpha| = \frac{1}{2}$.

We want to prove that we cannot get a function f of this form, using coefficients from the set K. We will use the asymptotics of the coefficients of f to prove this. Denoting the Taylor coefficients of $f(z)$ by g_n, it is sufficient to show that the product

$$|g_n| \cdot \sqrt{n!}$$

is not bounded between any two positive constants.

We first study the asymptotics of function of the form (1.1). Using Stirling's approximation we get

$$\sqrt{n!} = \left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right)\right)^{1/2} = (2\pi n)^{1/4}\left(\frac{n}{e}\right)^{n/2}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right). \tag{5.1}$$

The asymptotics of f are not as simple. Using rotation and scaling, we can assume $\alpha = \frac{1}{2}$ and $\gamma = 1$, moreover it is easy to see that β should not be zero. Therefore the problem is reduced to the study of the asymptotics of

$$\exp\left(\frac{1}{2} \cdot z^2 + \beta \cdot z\right) = \sum_{n=0}^{\infty} g_n(\beta) \cdot z^n, \tag{5.2}$$

with $\beta \neq 0$, with β possibly a complex number. A standard application of the saddle point method shows that

$$g_{n-1}(\beta) = C_\beta \cdot \left(\frac{e}{n}\right)^{\frac{n}{2}} \cdot \left(e^{\beta\sqrt{n}} + (-1)^n e^{-\beta\sqrt{n}}\right) \cdot \left(1 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right), \tag{5.3}$$

where C_β is some constant. We can see that this is not the same rate of decay as in (5.1) for $n \to \infty$. We arrive at the contradiction which finishes the proof of Theorem 2.

Remark. For the reader's convenience we prove the asymptotic estimate (5.3) in the appendix. We also note that the function (5.2) is the generating function of the Hermite polynomials.

6. Open Problems and Further Directions

Since it is known, that the expected number of zeros of the random function $f(z)$ of the form (1.1), in $r\mathbb{D}$ is r^2 it is interesting to get

logarithmic asymptotics for the probability of "large deviations"

$$\mathbb{P}\left(\left|n(r)-r^2\right| \geq \delta\cdot\sigma(r)^\alpha\right)$$

in terms of both α and δ (here $\sigma(r)$ denotes the standard deviation, and $\sigma(r)=\mathcal{O}\left(r^{1/2}\right)$). Some partial results in that direction can be found in Sodin and Tsirelson [ST3], Krishnapur [K] and Nazarov, Sodin and Volberg [NSV]. The most interesting cases here are $\alpha=2$ and $\alpha=4$. Note that the aforementioned results are based on the invariance of the random zero set and it would be also interesting to estimate similar probabilities for more general Gaussian entire functions.

Another possible direction would be to study the hole probability for more complicated domains. It would be interesting if there is a function, defined for closed (simply connected) domains U, denote it by $g(U)$, such that

$$\log\mathbb{P}\left(f(z)\neq 0 \text{ in } rU\right) = -g(U)\cdot r^4 + o(r^4).$$

Of course this function should be invariant with respect to the plane isometries (hence "geometrical").

7. Appendix: Proof of (5.3)

Here we compute the asymptotics of the Taylor coefficients of $\exp\left(\frac{1}{2}\cdot z^2+\alpha\cdot z\right)$ using the saddle point method. We write

$$\exp\left(\frac{1}{2}\cdot z^2+\alpha\cdot z\right)=\sum_{n=0}^{\infty} g_n(\alpha)\cdot z^n.$$

We begin with Cauchy's integral formula

$$g_{n-1}(\alpha)=\frac{1}{2\pi i}\int\limits_{r\mathbb{D}}\frac{\exp\left(\frac{1}{2}\cdot z^2+\alpha\cdot z\right)}{z^n}\,dz, \tag{7.1}$$

for some $r>0$ (since the function that we study is entire). We will use the notation (notice the use of n instead of $n-1$)

$$F_n(z)=F_n(\alpha;z)=\frac{1}{2}\cdot z^2+\alpha\cdot z-n\log z.$$

We use a standard saddle point method to study the asymptotics of this integral. We have

$$\frac{dF_n}{dz}=z+\alpha-\frac{n}{z},$$

for large values of n we see that the solutions to the equation $\frac{dF_n}{dz}=0$ are approximately $z_{1,2}=\pm\sqrt{n}$ (notice that they are the approximate

minima of the function F_n). Therefore we select the following rectangular contour ($T \gg 1$)

$$\begin{aligned}\Gamma &= \left[\sqrt{n} - i\cdot T, \sqrt{n} + i\cdot T\right] \bigcup \left[-\sqrt{n} - i\cdot T, -\sqrt{n} + i\cdot T\right] \\ &\quad \bigcup \left[-\sqrt{n} - i\cdot T, \sqrt{n} - i\cdot T\right] \bigcup \left[-\sqrt{n} + i\cdot T, \sqrt{n} + i\cdot T\right] \\ &= \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4.\end{aligned}$$

We have the following estimate for F_n ($z = x + iy$)

$$|\exp\left(F_n\left(z\right)\right)| \le \exp\left(\frac{x^2 - y^2}{2} + |\alpha| \cdot |z|\right) \cdot |z|^{-n}. \tag{7.2}$$

We see that for $T \gg T_0(\alpha, n)$, n fixed, we have

$$|\exp\left(F_n\left(z\right)\right)| \le \exp\left(-T^2/4\right)$$

and so

$$\left|\frac{1}{2\pi i} \int\limits_{\Gamma_3 \cup \Gamma_4} \exp\left(F_n(z)\right)\, dz\right| \le C\sqrt{n} \cdot \exp\left(-T^2/4\right) \to 0,$$

as $T \to \infty$. Therefore what is left is to estimate the integrals

$$\begin{aligned}I_1 &= \frac{1}{2\pi i} \cdot \int\limits_{\sqrt{n} + i\mathbb{R}} \exp\left(F_n(z)\right)\, dz, \\ I_2 &= \frac{1}{2\pi i} \cdot \int\limits_{-\sqrt{n} - i\mathbb{R}} \exp\left(F_n(z)\right)\, dz.\end{aligned}$$

We start by using the parametrization $z = \sqrt{n} \cdot (1 + i \cdot t)$, with $t \in (-\infty, \infty)$. Then we have

$$\begin{aligned}I_1 &= \frac{n^{1/2}}{2\pi} \cdot \int\limits_{-\infty}^{\infty} \exp\left(\frac{n}{2} \cdot (1 + i\cdot t)^2 + \alpha\sqrt{n} \cdot (1 + i \cdot t)\right) \cdot n^{-n/2} \cdot (1 + i \cdot t)^{-n}\, dt \\ &= \frac{1}{2\pi} \cdot \frac{\exp\left(\frac{n}{2} + \alpha\sqrt{n}\right)}{n^{(n-1)/2}} \cdot \int\limits_{-\infty}^{\infty} \exp\left(-\frac{n}{2} \cdot t^2 + \left(n + \alpha\sqrt{n}\right) \cdot ti\right) \cdot (1 + i \cdot t)^{-n}\, dt.\end{aligned}$$

We denote the new integral by I_1'. Using (7.2) we want to show that only the "small" values of t contribute to the asymptotics.

Set $0 < b < \frac{1}{2}$, we want b to be sufficiently close to $\frac{1}{2}$ and to have

$$-\frac{n}{2} \cdot t^2 + \sqrt{n}\,|\alpha| \cdot |t| \le -b \cdot nt^2,$$

or after rearrangement

$$|t| \geq \frac{|\alpha|}{\sqrt{n}} \cdot \frac{1}{\frac{1}{2} - b}.$$

We assume $n \geq 2^{24}$ and select $b = \frac{1}{2} - \frac{1}{n^{1/12}}$ (the reason for selecting $\frac{1}{12}$ will be clear from the calculations bellow). For $|t| \geq \frac{|\alpha|}{n^{5/12}}$, we now have

$$\begin{aligned}
\left|\exp\left(-\frac{n}{2} \cdot t^2 + \left(n + \alpha\sqrt{n}\right) \cdot ti\right)\right| &\leq \exp\left(-\frac{n}{2} \cdot t^2 + \sqrt{n}\,|\alpha| \cdot |t|\right) \\
&\leq \exp\left(-\left(\frac{1}{2} - \frac{1}{n^{1/12}}\right) \cdot nt^2\right) \\
&\leq \exp\left(-\frac{1}{4} \cdot nt^2\right).
\end{aligned}$$

We also note that for n large enough

$$\left|1 + i \cdot \frac{|\alpha|}{n^{5/12}}\right|^n \geq \left(1 + \frac{|\alpha|^2}{n^{5/6}}\right)^{n/2} \geq \exp\left(\frac{1}{3} \cdot |\alpha|^2 n^{1/6}\right).$$

Therefore we have the following estimate for "large" values of t

$$\begin{aligned}
&\left|\int\limits_{|t| \geq \frac{|\alpha|}{n^{5/12}}} \exp\left(-\frac{n}{2} \cdot t^2 + \left(n + \alpha\sqrt{n}\right) \cdot ti\right) \cdot (1 + i \cdot t)^{-n}\, dt\right| \leq \\
&\qquad \int\limits_{|t| \geq \frac{|\alpha|}{n^{5/12}}} \exp\left(-\frac{n}{4} \cdot t^2\right) dt \cdot \left|1 + i \cdot \frac{|\alpha|}{n^{5/12}}\right|^{-n} \leq \\
&\qquad \int\limits_{-\infty}^{\infty} \exp\left(-\frac{n}{4} \cdot t^2\right) dt \cdot \exp\left(-\frac{1}{3} \cdot |\alpha|^2 n^{1/6}\right) \leq \\
&\qquad \exp\left(-\frac{1}{3} \cdot |\alpha|^2 n^{1/6}\right).
\end{aligned}$$

For the main part we use the Taylor expansion of $\log(1 + x)$ for small x,

$$\log(1 + i \cdot t) = i \cdot t + \frac{t^2}{2} + \sum_{m=3}^{\infty} (-1)^{m+1} \cdot \frac{(it)^m}{m}.$$

For $|t| < \frac{|\alpha|}{n^{5/12}}$ and $n^{5/12} \geq 2|\alpha|$, we have the following bound for the tail

$$\left|\sum_{m=3}^{\infty} (-1)^{m+1} \cdot \frac{(it)^m}{m}\right| \leq |t|^3 \cdot \left|\sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m\right| = 2|t|^3,$$

so we get

$$\left|\log(1 + i \cdot t) - \left(i \cdot t + \frac{t^2}{2}\right)\right| \leq 2|t|^3,$$

or

$$\left|n \cdot \log(1 + i \cdot t) - n\left(i \cdot t + \frac{t^2}{2}\right)\right| \leq 2n|t|^3.$$

Using $\exp(n|t|^3) = 1 + n|t|^3 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$, we now have

$$I_1' = I_1'' + I_1^{(1)} + \mathcal{O}\left(\frac{1}{\sqrt{n}} \cdot I_1^{(2)}\right).$$

where

$$\begin{aligned}
I_1'' &= \int_{|t|<\frac{|\alpha|}{n^{5/12}}} \exp\left(-nt^2 + \alpha\sqrt{n} \cdot ti\right) dt,\\
I_1^{(1)} &= \int_{|t|<\frac{|\alpha|}{n^{5/12}}} \exp\left(-nt^2 + \alpha\sqrt{n} \cdot ti\right) \cdot 2n|t|^3 \, dt,\\
I_1^{(2)} &= \int_{|t|<\frac{|\alpha|}{n^{5/12}}} \exp\left(-nt^2\right) dt.
\end{aligned}$$

After simple approximations of $I_1^{(1)}$ and $I_1^{(2)}$ we get

$$I_1' = I_1'' + \mathcal{O}\left(\frac{1}{n}\right).$$

Using similar estimates we conclude

$$\left|I_1'' - \int_{-\infty}^{\infty} \exp\left(-nt^2 + \alpha\sqrt{n} \cdot ti\right) dt\right| \leq \int_{|t|\geq\frac{|\alpha|}{n^{5/12}}} \exp\left(-\frac{n}{2} \cdot t^2\right) dt,$$

the error term is bounded in the following way

$$\int\limits_{|t|\geq\frac{|\alpha|}{n^{5/12}}} \exp\left(-\frac{n}{2}\cdot t^2\right)\,dt \;\leq\; \int\limits_{|t|\geq\frac{|\alpha|}{n^{5/12}}} \exp\left(-\frac{n}{2}\cdot t^2\right)\cdot|t|\cdot\frac{n^{5/12}}{|\alpha|}\,dt$$
$$= \frac{2\exp\left(-\frac{|\alpha|^2}{2}\cdot n^{1/6}\right)}{|\alpha|n^{7/12}}.$$

Overall we have (for n large enough)

$$I_1' = \int\limits_{-\infty}^{\infty} \exp\left(-nt^2+\alpha\sqrt{n}\cdot ti\right)\,dt + \mathcal{O}\left(\frac{1}{n}\right) = \sqrt{\frac{\pi}{n}}\cdot\exp\left(-\alpha^2/4\right)+\mathcal{O}\left(\frac{1}{n}\right),$$

and so

$$I_1 = \sqrt{\frac{1}{4\pi}}\cdot\frac{\exp\left(\frac{n}{2}+\alpha\sqrt{n}-\alpha^2/4\right)}{n^{n/2}}\cdot\left(1+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right).$$

Repeating the same calculation for I_2 we also have

$$I_2 = \sqrt{\frac{1}{4\pi}}\cdot(-1)^n\cdot\frac{\exp\left(\frac{n}{2}-\alpha\sqrt{n}-\alpha^2/4\right)}{n^{n/2}}\cdot\left(1+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right).$$

Recalling (7.1)

$$g_{n-1}(\alpha) = C_\alpha\cdot\left(\frac{e}{n}\right)^{\frac{n}{2}}\cdot\left(e^{\alpha\sqrt{n}}+(-1)^n e^{-\alpha\sqrt{n}}\right)\cdot\left(1+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right).$$

Comparing this with (5.1) we get the required contradiction.

References

[Ahl] L. V. Ahlfors, Complex Analysis, 3rd Edition. McGraw Hill, NY, 1979.

[BH] J. Ben Hough, Large deviations for the zero set of an analytic function with diffusing coefficients. E print Archive, arXiv:math/0510237, (2008).

[BKPV] J. Ben Hough, M. Krishnapur, Y. Peres and B. Vir\'{a}g, Zeros of Gaussian analytic functions and Determinantal point processes. To be published by AMS.

[K] M. Krishnapur, Overcrowding estimates for zeroes of Planar and Hyperbolic Gaussian analytic functions. J. of Stat. Phys., **124**, no. 6, (2006), 1399–1423.

[Lev] B. Ya. Levin, Lectures on Entire Functions. American Mathematical Society, Translations of mathematical monographs v.**150**, 1996.

[NSV] F. Nazarov, M. Sodin, and A. Volberg, The Jancovici–Lebowitz–Manificat Law for Large Fluctuations of Random Complex Zeroes. Commun. Math. Phys. **284** (2008), 833–865.

[ST1] M. Sodin and B. Tsirelson, Random complex zeroes, I. Asymptotic normality. Israel J. Math. **144** (2004), 125–149.

[ST3] M. Sodin and B. Tsirelson, Random Complex zeroes. III. Decay of the hole probability. Israel J. Math **147** (2005), 371–379.

[SZZ] B. Shiffman, S. Zelditch, S. Zrebiec, Overcrowding and hole probabilities for random zeros on complex manifolds. Indiana Univ. Math. J. **57** (2008), 1977–1997.

[Zr1] S. Zrebiec, The zeros of flat Gaussian random holomorphic functions on $\mathbb{C}^n$, and hole probability. Michigan Math. J., **55** (2007), no. 2, 269–284.

[Zr2] S. Zrebiec, The order of the decay of the hole probability for Gaussian random $SU(m+1)$ polynomials. E print Archive, arXiv::math.CV/0704.2733, (2007).